THE MODULI SPACE OF 6-DIMENSIONAL 2-STEP NILPOTENT LIE ALGEBRAS

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Abstract. We determine the moduli space of metric 2-step nilpotent Lie algebras of dimension up to 6. This space is homeomorphic to a cone over a 4-dimensional contractible simplicial complex.

Keywords: nilpotent Lie algebra, moduli space

1. Introduction

The geometry of 2-step nilpotent Lie groups with a left-invariant metric is very rich and has been widely studied since the papers of A. Kaplan [9, 10] and P. Eberlein [4] (see, for instance [2, 3, 5, 6, 7, 12, 15] for recent papers on this subject). Important examples of such Lie groups are provided by groups of Heisenberg type [10].

In general, the moduli space of metric Lie algebras of a fixed dimension is a cone with peak the abelian Lie algebra and basis the subset obtained by normalizing the Lie bracket c, for instance requiring $\mathrm{Tr}(c^*c) = 2$. In the present paper we determine the moduli space $\mathcal{N}_6$ of 6-dimensional 2-step nilpotent Lie algebras endowed with a metric. We show that $\mathcal{N}_6$ is a cone over an explicitely given contractible 4-dimensional simplicial complex. We also exhibit standard metric representatives of the 7 isomorphism types of 6-dimensional 2-step nilpotent Lie algebras within our picture. This contains all deformations of these Lie algebras, cf. [12].

In [12] J. Lauret identified in a natural way each point of the variety of real Lie algebras with a left-invariant Riemannian metric on a Lie group and studied the interplay between invariant-theoretic and Riemannian aspects of this variety. We show that on a certain subset of $\mathcal{N}_6$ the nullity of the Riemannian curvature tensor singles out products.

1991 *Mathematics Subject Classification.* Primary 22E25, 53C30, 22E60.

* Research partially supported by MURST, GNSAGA (Indam) of Italy and EDGE Research Training Network HPRN-CT-2000-00101.

** Research supported by GNSAGA (INDAM) of Italy and the University of Cyprus.

The subspace $\mathcal{N}_{n,k} \subset \mathcal{N}_n$ of Lie algebras with k-dimensional commutator ideal contains the subspace $\mathcal{D}_{n,k}$ of algebras with isometric c^* as a strong deformation retract. For the algebras $\mathfrak{n}_{(\alpha_+,\alpha_-)} \in \mathcal{D}_{6,2}$ we give the structure equations, write down the curvature tensor and compute their infinitesimal rank, i.e. the minimal nullity of the Jacobi operators. In [15] it was proved that groups of Heisenberg type have infinitesimal rank one. We show that this is also the case for any $\mathfrak{n}_{(\alpha_+,\alpha_-)} \in \mathcal{D}_{6,2}$ with the exception of $\mathfrak{n}_{(1,1)} \cong \mathfrak{h}_3 \oplus \mathfrak{h}_3$ and $\mathfrak{n}_{(1/2,1/2)} \cong \mathfrak{n}_5 \oplus \mathbb{R}$, both endowed with the product metric, whose rank is two.

2. Preliminaries

A Lie algebra $\mathfrak{g}$ is nilpotent if its central series ends, i.e. in the sequence of ideals of $\mathfrak{g}$ recursively defined by $\mathfrak{g}^0 := \mathfrak{g}$, $\mathfrak{g}^{i+1} := [\mathfrak{g}, \mathfrak{g}^i]$ there is an integer k such that $\mathfrak{g}^k = 0$. Then $\mathfrak{g}$ is a k-step nilpotent if $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$. Thus a 2-step nilpotent Lie algebra $\mathfrak{n}$ is a Lie algebra such that its commutator ideal $\mathfrak{n}^1 := [\mathfrak{n}, \mathfrak{n}]$ is contained in its centre.

A left-invariant metric on a (simply connected) 2-step nilpotent Lie group N is given by a scalar product $\langle \cdot, \cdot \rangle$ on its Lie algebra $\mathfrak{n}$. We will call such a Lie algebra "metric 2-step nilpotent Lie algebra".

A simply connected 2-step nilpotent Lie group with left-invariant metric is uniquely determined by the triple $(\mathfrak{h}, \mathfrak{z}, j)$ [4, 9], where $\mathfrak{h}$ and $\mathfrak{z}$ are real vector spaces with positive definite scalar product and $j\colon \mathfrak{z} \to \mathfrak{so}(\mathfrak{h})$ is the homomorphism (of vector spaces, not necessarily of Lie algebras) related to the Lie bracket by

$$\langle [x,y], z \rangle = \langle y, j(z)x \rangle \quad \forall x, y \in \mathfrak{h}, z \in \mathfrak{z} \ .$$

Thus, j is essentially the adjoint of the Lie bracket $c\colon \Lambda^2\mathfrak{h} \to \mathfrak{z}$. Requiring in addition j to be injective makes this correspondence one to one.

Observe that if one identifies $\mathfrak{z}$ with its dual $\mathfrak{z}^*$ via the metric (and similarly for $\mathfrak{h}$ and $\mathfrak{n}$), then the differential $d : \Lambda^1\mathfrak{z}^* \subset \Lambda^1\mathfrak{n}^* \to \Lambda^2\mathfrak{h}^* \subset \Lambda^2\mathfrak{n}^*$ can be identified with j and $\dim(\operatorname{Im} d) = \dim(\mathfrak{n}^1)$.

By [13, 8] there are 34 classes of 6-dimensional nilpotent Lie algebras. Out of these 34 classes, the 2-step nilpotent have the following structure

equations

$$\begin{array}{l}(0,0,0,12,13,23)\\(0,0,0,0,13+42,14+23)=\mathfrak{h}_3^{\mathbb{C}},\\(0,0,0,0,12,14+23),\\(0,0,0,0,12,34)=\mathfrak{h}_3\oplus\mathfrak{h}_3,\\(0,0,0,0,12,13)=\mathfrak{n}_5\oplus\mathbb{R},\\(0,0,0,0,0,12+34)=\mathfrak{h}_5\oplus\mathbb{R},\\(0,0,0,0,0,12)=\mathfrak{h}_3\oplus\mathbb{R}^3,\end{array}$$

where $\mathfrak{h}_3^{\mathbb{C}}$ is the complex 3-dimensional Heisenberg Lie algebra, $\mathfrak{h}_3$ the real 3-dimensional Heisenberg Lie algebra and $\mathfrak{n}_5=(0,0,0,12,13)$. We use the notation of [14]. For example, $(0,0,0,12,13,23)$ denotes the Lie algebra with $de^i=0, i=1,\dots 3$, $de^4=e^1\wedge e^2$, $de^5=e^1\wedge e^3$, $de^6=e^2\wedge e^3$, where (e^j) is a basis of left-invariant 1-forms.

3. The Moduli space of 2-step nilpotent Lie algebras

We now determine the moduli space of metric 2-step nilpotent Lie algebras. Let $\mathfrak{g}$ be a metric Lie algebra of dimension n. We can always choose a linear isometry of $\mathfrak{g}$ with Euclidean space $\mathbb{R}^n$ (endowed with its standard scalar product). The set of Lie brackets on $\mathbb{R}^n$ is an algebraic subset of $\mathrm{Hom}(\Lambda^2\mathbb{R}^n,\mathbb{R}^n)$, whose ideal is given by the Jacobi identity, i.e.,

$$\begin{aligned}\widehat{\mathcal{L}}_n := \; & \{c\in \mathrm{Hom}(\Lambda^2\mathbb{R}^n,\mathbb{R}^n) \mid \\ & c(c(u,v),w)+c(c(w,u),v)+c(c(v,w),u)=0 \;\forall u,v,w\in\mathbb{R}^n\}\ .\end{aligned}$$

The set of 2-step nilpotent Lie brackets on $\mathbb{R}^n$ is

$$\widehat{\mathcal{N}}_n := \{c\in \mathrm{Hom}(\Lambda^2\mathbb{R}^n,\mathbb{R}^n) \mid c(c(u,v),w)=0 \;\forall\; u,v,w\in\mathbb{R}^n\}\ .$$

These sets are invariant under the $\mathrm{GL}(n,\mathbb{R})$-action on $\mathrm{Hom}(\Lambda^2\mathbb{R}^n,\mathbb{R}^n)$. The moduli space of 2-step nilpotent (metric) n-dimensional Lie algebras is the space of (isometric) isomorphism classes of such Lie algebras. It inherits its topology as the quotient of $\widehat{\mathcal{N}}_n$ by the action of $\mathrm{GL}(n,\mathbb{R})$ (respectively, $\mathrm{O}(n)$),

$$\mathcal{N}_n := \widehat{\mathcal{N}}/\mathrm{O}(n) \qquad (\text{resp. } \check{\mathcal{N}}_n=\widehat{\mathcal{N}}/\mathrm{GL}(n,\mathbb{R}))\ .$$

For $k\le n$ we decompose $\mathbb{R}^n=\mathbb{R}^{n-k}\oplus\mathbb{R}^k$ orthogonally. A metric n-dimensional 2-step nilpotent Lie algebra $\mathfrak{n}$ with $\dim\mathfrak{n}^1=k$ is isometric to $(\mathbb{R}^n,c)$ where c is a 2-step nilpotent Lie bracket of rank $k=\dim(\mathrm{Im}c)$ such that $\mathrm{Im}c=\{0\}\oplus\mathbb{R}^k$. We define

$$\widehat{\mathcal{N}}_{n,k} := \{c\in \mathrm{Hom}(\Lambda^2\mathbb{R}^{n-k},\mathbb{R}^k) \mid c \text{ surjective}\}.$$

This space carries an action of $\mathrm{GL}(n-k,\mathbb{R})\times\mathrm{GL}(k,\mathbb{R})$. The moduli space of n-dimensional (metric) 2-step nilpotent Lie algebras with k-dimensional commutator ideal is the quotient

$$\begin{aligned}&\mathcal{N}_{n,k}=\widehat{\mathcal{N}}_{n,k}/(\mathrm{O}(n-k)\times\mathrm{O}(k))\\&(\text{resp. } \check{\mathcal{N}}_{n,k}=\widehat{\mathcal{N}}_{n,k}/(\mathrm{GL}(n-k,\mathbb{R})\times\mathrm{GL}(k,\mathbb{R}))).\end{aligned}$$

Extending $c\in\widehat{\mathcal{N}}_{n,k}$ by 0 to all of $\Lambda^2\mathbb{R}^n$, we may view $\widehat{\mathcal{N}}_{n,k}\subset\widehat{\mathcal{N}}_n$ and decompose

$$\mathcal{N}_n=\bigcup_{0\le k\le\binom{n-k}{2}}\mathcal{N}_{n,k}\qquad\left(\text{resp. } \check{\mathcal{N}}_n=\bigcup_{0\le k\le\binom{n-k}{2}}\check{\mathcal{N}}_{n,k}\right). \tag{3.1}$$

We denote by $\gamma_{k,V}\to\mathrm{Gr}_k(V)$ the tautological vector bundle over the Grassmanian of k-planes in a real vector space V. We let $\mathsf{S}^2_+\gamma^*_{k,V}\subset\mathsf{S}^2\gamma^*_{k,V}$ be the set of positive definite symmetric 2-tensors on $\gamma_{k,V}$.

The adjoint c^* of $c\in\widehat{\mathcal{N}}_{n,k}$ is injective on $\mathbb{R}^k$. Pushing forward the standard scalar product g_{std} on $\mathbb{R}^k$, a scalar product on its image is defined. The maps

$$\begin{array}{ccccc}\widehat{\mathcal{N}}_{n,k} & \xrightarrow{\phi} & \mathsf{S}^2_+\gamma^*_{k,\Lambda^2\mathbb{R}^{n-k}} & \xrightarrow{\pi} & \mathrm{Gr}_k(\Lambda^2\mathbb{R}^{n-k})\\ c & \mapsto & (\mathrm{Im}c^*, c^*g_{\mathrm{std}}) & \mapsto & \mathrm{Im}c^*\end{array} \tag{3.2}$$

are $\mathrm{O}(n-k)\times\mathrm{O}(k)$-equivariant. In particular

Theorem 3.1. *There is a homeomorphism*

$$\mathcal{N}_{n,k}\approx\mathsf{S}^2_+\gamma^*_{k,\Lambda^2\mathbb{R}^{n-k}}/\mathrm{O}(n-k)$$

and a strong deformation retraction

$$\mathcal{N}_{n,k}\simeq\mathrm{Gr}_k(\Lambda^2\mathbb{R}^{n-k})/\mathrm{O}(n-k)=:\mathcal{D}_{n,k}.$$

Here $\mathcal{D}_{n,k}\hookrightarrow\mathcal{N}_{n,k}$ is identified with the subset of those 2-step nilpotent Lie algebras with isometric $j=c^*\colon\mathbb{R}^k\to\mathfrak{so}(n-k)$.

Proof. Two Lie brackets $c,c'\in\widehat{\mathcal{N}}_{n,k}\subset\mathrm{Hom}(\Lambda^2\mathbb{R}^{n-k},\mathbb{R}^k)$ are equivalent in $\mathcal{N}_{n,k}$ if there are $A\in\mathrm{O}(k)$ and $T\in\mathrm{O}(n-k)$ such that $AcT^{-1}=c'$. Equivalent formulations are

$$\begin{gathered}(T^{-1})^*c^*A^*=c'^*,\\ c^*A^*\left(c'^*|_{\mathrm{Im}c'^*}\right)^{-1}=T^*,\end{gathered}$$

that is to say, $T^*\colon\mathrm{Im}c'^*\to\mathrm{Im}c^*$ is isometric with respect to the metrics pushed forward by c,c'. Thus, the map $\widehat{\mathcal{N}}_{n,k}\xrightarrow{\phi}\mathsf{S}^2_+\gamma^*_{k,\Lambda^2\mathbb{R}^{n-k}}$ in (3.2) induces an homeomorphism on both quotients by $\mathrm{O}(n-k)\times\mathrm{O}(k)$.

Let g_0 be an $\mathrm{O}(n-k)$-invariant scalar product on $\Lambda^2\mathbb{R}^{n-k}$ (for instance the opposite of the Cartan-Killing form on $\Lambda^2\mathbb{R}^{n-k}=\mathfrak{so}(n-k)$).

Then a homotopy inverse to π is given by $\sigma\colon U \in \mathrm{Gr}_k(\Lambda^2\mathbb{R}^{n-k}) \mapsto (U, g_0|_U)$. Clearly, $\pi\circ\sigma = \mathrm{id}_{\mathrm{Gr}_k(\Lambda^2\mathbb{R}^{n-k})}$ and $H((U,g),t) := (U,(1-t)g + tg_0|_U)$ defines an $\mathrm{O}(n-k)$-equivariant homotopy $\sigma\circ\pi \simeq \mathrm{id}_{\mathsf{S}^2_+\gamma^*_{k,\Lambda^2\mathbb{R}^{n-k}}}$. $\square$

Remark 1. Using Theorem 3.1, one can describe some special cases for $\mathcal{N}_{n,k}$.

• For $k=0$, $\mathcal{N}_{n,0}$ is a point, corresponding to the n-dimensional abelian Lie algebra.

• For $k=1$, $\mathrm{Gr}_1(\Lambda^2\mathbb{R}^{n-1})$ is homeomorphic to the real projective space $\mathbb{R}\mathrm{P}^{\binom{n-1}{2}-1}$. Moreover, $A \in \Lambda^2\mathbb{R}^{n-1} \cong \mathfrak{so}(n-1)$ is conjugate to a block-diagonal matrix with 2×2 blocks $\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$ on the diagonal and such that $0 \le \lambda_1 \le \ldots \le \lambda_{[\frac{n-1}{2}]}$. Hence $\mathrm{Gr}_1(\Lambda^2\mathbb{R}^{n-1})/\mathrm{O}(n-1) \approx \Delta^{[\frac{n-1}{2}]-1}$ is homeomorphic to a $([\frac{n-1}{2}]-1)$-simplex and

$$\mathcal{N}_{n,1} \approx \Delta^{[\frac{n-1}{2}]-1} \times \mathbb{R}^+ .$$

For odd n and $\lambda_1 = \ldots = \lambda_{\frac{n-1}{2}} = 1$ we recover the n-dimensional Heisenberg algebras $\mathfrak{h}_n \in \mathcal{N}_{n,1}$.

• If $k = \binom{n-k}{2}$, then $\mathrm{Gr}_k(\Lambda^2\mathbb{R}^{n-k})$ is homeomorphic to a point and $\mathcal{N}_{n,k} \approx \mathsf{S}^2_+(\Lambda^2\mathbb{R}^{n-k})^*/\mathrm{O}(n-k)$ is a quotient of the cone $\mathsf{S}^2_+(\Lambda^2\mathbb{R}^{n-k})^*$.

4. Metric 2-step nilpotent Lie algebras of dimension ≤ 6

In this section we study in detail the case of Lie algebras of dimension up to 6. We denote by $\mathcal{N}^0_{*,*}$ the subspace of Lie algebras with $\mathrm{Tr}(j^*j) = 2$. The whole space $\mathcal{N}_*$ is a cone over $\mathcal{N}^0_*$ whose peak is the abelian Lie algebra. Clearly, for $m \le 2$, $\mathcal{N}_m$ is a point, the abelian Lie algebra. For $m = 3, 4$ we get $\mathcal{N}^0_3 = \{\mathfrak{h}_3\}$ and $\mathcal{N}^0_4 = \{\mathfrak{h}_3 \oplus \mathbb{R}\}$. For $m = 5$ we have $\mathcal{N}_5 = \mathcal{N}_{5,0} \cup \mathcal{N}_{5,1} \cup \mathcal{N}_{5,2}$. By remark 1, $\mathcal{N}^0_{5,1}$ is homeomorphic to an interval with endpoints the Lie algebras $\mathfrak{h}_3 \oplus \mathbb{R}^2$ and $\mathfrak{h}_5$. Let now $\mathfrak{n}_5 \in \mathcal{N}_{5,2}$ denote a Lie algebra with isometric $j\colon \mathbb{R}^2 \to \mathfrak{so}(3)$; all such Lie algebras are isometrically isomorphic. We will se later that the closure $\overline{\mathcal{N}^0_{5,2}}$ is homeomorphic to an interval with endpoints $\mathfrak{h}_3 \oplus \mathbb{R}^2$ and $\mathfrak{n}_5$. For any $m \le n$ there are embeddings $\mathcal{N}_{m,k} \hookrightarrow \mathcal{N}_{n,k}$, $\mathfrak{n} \mapsto \mathfrak{n} \oplus \mathbb{R}^{n-m}$. Thus all the spaces of Lie algebras above appear in $\mathcal{N}_6$.

In the sequel we will show that $\mathcal{N}_6$ is a cone over a contractible 4-dimensional simplicial complex pictured in Figure 1. The decomposition (3.1) becomes

$$\mathcal{N}_6 = \mathcal{N}_{6,0} \cup \mathcal{N}_{6,1} \cup \mathcal{N}_{6,2} \cup \mathcal{N}_{6,3} .$$

From remark 1 we have

$$\begin{aligned}\mathcal{N}_{6,0} &\approx * \ ,\\ \mathcal{N}_{6,1} &\approx [0,1]\times\mathbb{R}^+ \ ,\\ \mathcal{N}_{6,3} &\approx \mathsf{S}^2_+(\Lambda^2\mathbb{R}^3)^*/\mathrm{O}(3) \ .\end{aligned}$$

4.1. **Invariants for $\mathcal{N}_6$.** The subsequent simultaneous description of the pieces of (3.1) and their glueing relies on the isomorphism of Lie algebras

$$\mathfrak{so}(4) \cong \mathfrak{so}(3)\oplus\mathfrak{so}(3) = \mathfrak{su}_+(2)\oplus\mathfrak{su}_-(2) = \mathbb{R}^3_+\oplus\mathbb{R}^3_- \ . \tag{4.3}$$

Under the identification (4.3) the action of $\mathrm{SO}(4)$ on $\mathfrak{so}(4)$ translates to the (dual of the) usual action of $\mathrm{SO}(3)\times\mathrm{SO}(3)$ on $\mathbb{R}^3_+\oplus\mathbb{R}^3_-$. The whole orthogonal group in addition contains an element $\tau\in\mathrm{O}(4)$ of determinant -1 which interchanges the factors. Explicitely, the isomorphism (4.3) is given by mapping

$$\xi e_1^\pm + \psi e_2^\pm + \chi e_3^\pm = \begin{pmatrix} i\xi & \psi + i\chi \\ -\psi + i\chi & -i\xi \end{pmatrix} \in \mathfrak{su}_\pm(2)$$

to

$$\begin{pmatrix} 0 & \xi & \psi & \chi \\ -\xi & 0 & -\chi & \psi \\ -\psi & \chi & 0 & -\xi \\ -\chi & -\psi & \xi & 0 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 & \xi & \psi & \chi \\ -\xi & 0 & \chi & -\psi \\ -\psi & -\chi & 0 & \xi \\ -\chi & \psi & -\xi & 0 \end{pmatrix}$$

for "$-$" and "$+$" respectively. The diagonal matrix $\mathrm{diag}(-1,1,1,1)$ acts as the involution τ. We denote the two components of j by $j_\pm\colon \mathbb{R}^2\to\mathbb{R}^3_\pm$. The spectra of $j_\pm^* j_\pm$ and the trace of $j_-^* j_- j_+^* j_+$ are invariant under the $\mathrm{O}(2)\times\mathrm{O}(4)$-action, up to interchanging $\pm$.

We claim that these data suffice to determine the equivalence class of j under the $\mathrm{O}(2)\times\mathrm{O}(4)$-action:

Clearly, the entire matrices $j_-^*j_-$ and $j_+^*j_+$ determine j up to the action of $\mathrm{O}(4)$. If both $j_-^*j_-$ and $j_+^*j_+$ have two identical eigenvalues, then both matrices are diagonal for any orthonormal basis of $\mathbb{R}^2$. Otherwise, after possibly using τ to permute $\pm$, we may assume that $j_-^*j_-$ has two different eigenvalues α_-, β_- and that e_1, e_2 are the respective eigenvectors. If $j_+^*j_+ = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$, then $\mathrm{Tr}(j_+^*j_+) = x+y$, $\det(j_+^*j_+) = xy - z^2$ and $\mathrm{Tr}(j_+^*j_+j_-^*j_-) = \alpha_- x + \beta_- y$ determine $x, y \geq 0$ and z up to sign. Since the sign of z can be changed by conjugation with $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we may assume $z\geq 0$. Thus, all of $j_+^*j_+$ is determined by the above invariants.

Let $\mathrm{Spec}(j_\pm^* j_\pm) = \{\alpha_\pm, \beta_\pm\}$, with $0 \le \alpha_\pm \le \beta_\pm$ and $\mathrm{Tr}(j_+^* j_+ j_-^* j_-) = t$. The possible range for t in dependence of $\alpha_\pm, \beta_\pm$ is obtained by solving

$$t = \alpha_- x + \beta_- y \ , \quad x + y = \alpha_+ + \beta_+ \ , \quad xy - z^2 = \alpha_+\beta_+ \ , \quad z \ge 0 \ .$$

We get

$$t \in I_{\alpha_\pm,\beta_\pm} = [\alpha_-\beta_+ + \alpha_+\beta_-, \alpha_-\alpha_+ + \beta_-\beta_+] \ .$$

Let S denote the set of all 5-tuples $(\alpha_\pm, \beta_\pm, t)$ satisfying the above conditions and subject to the relation induced from τ, i.e.

$$S := \left\{ (\alpha_-, \beta_-, \alpha_+, \beta_+, t) \in \mathbb{R}^5 \;\middle|\; \begin{array}{l} 0 \le \alpha_- \le \beta_-, \\ 0 \le \alpha_+ \le \beta_+, \\ t \in I_{\alpha_\pm,\beta_\pm} \end{array} \right\} \Big/ \sim$$

with the identification

$$(\alpha_-, \beta_-, \alpha_+, \beta_+, t) \sim (\alpha_+, \beta_+, \alpha_-, \beta_-, t) \ .$$

We have

Theorem 4.1. *The closure of $\mathcal{N}_{6,2}$ is homeomorphic to S under the map*

$$\begin{array}{rcl} \Psi \colon \overline{\mathcal{N}_{6,2}} & \to & S \\ j = (j_-, j_+) & \mapsto & (\mathrm{Spec}(j_-^* j_-), \mathrm{Spec}(j_+^* j_+), \mathrm{Tr}(j_+^* j_+ j_-^* j_-)) \end{array}$$

Proof. We have already shown that the above map is bijective. It is continuous since the spectrum of a matrix depends continuosly on its entries. Since

$$\mathrm{Tr}(j^* j) = \mathrm{Tr}(j_-^* j_-) + \mathrm{Tr}(j_+^* j_+) = \alpha_-(j) + \beta_-(j) + \alpha_+(j) + \beta_+(j),$$

we get that, for all $r > 0$, Ψ defines a continuous bijection

$$\begin{array}{l} \{j \mid \mathrm{Tr}(j^* j) \le r\}/_{\mathrm{O}(2)\times\mathrm{O}(4)} \leftrightarrow \\ S \cap \{(\alpha_-, \beta_-, \alpha_+, \beta_+, t) \in \mathbb{R}^5 \mid \alpha_- + \beta_- + \alpha_+ + \beta_+ \le r\}/_\sim \end{array}$$

which is a homeomorphism since these sets are compact. It follows that Ψ is a homeomorphism. □

The spaces $\mathcal{N}_{6,1}$ and $\mathcal{N}_{6,3}$ are treated similarly. For $\mathcal{N}_{6,1}$ we have to deal with maps $j \colon \mathbb{R} \to \mathfrak{so}(5)$. Any such map is conjugate to some $j \colon \mathbb{R} \to \mathfrak{so}(4) \subset \mathfrak{so}(5)$. Extending j by 0 to a map $\mathbb{R}^2 \to \mathfrak{so}(4)$, we can identify $\mathcal{N}_{6,1}$ with a subset of $\partial\overline{\mathcal{N}_{6,2}}$. In the terminology above, both components j_+ and j_- have only one nonvanishing eigenvalue, $0 \le \beta_+ \le \beta_-$ respectively. Moreover $\mathrm{Tr}(j_+^* j_+ j_-^* j_-) = \beta_-\beta_+$ gives no new invariant on $\mathcal{N}_{6,1}$.

For $\mathcal{N}_{6,3}$, we observe that the imbedding $\mathfrak{so}(3) \hookrightarrow \mathfrak{so}(4)$ induced from $\mathbb{R}^3 \hookrightarrow \mathbb{R}^4$ translates under (4.3) to the skew-diagonal map $\mathfrak{so}(3) \ni X \mapsto$

$\frac{1}{2}(X, -X) \in \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{so}(4)$. Thus, for $j = (j_-, j_+) \in \mathcal{N}_{6,3}$ we have $j_+ = -j_-$. Hence, $\mathrm{Spec}(j_-^* j_-) = \mathrm{Spec}(j_+^* j_+) = \{\omega, \alpha, \beta\}$ with $0 \leq \omega \leq \alpha = \alpha_- = \alpha_+ \leq \beta = \beta_- = \beta_+$ and $t = \omega^2 + \alpha^2 + \beta^2$.

As a whole, $\mathcal{N}_6$ is a cone over the set $\mathcal{N}_6^0$ of those isometric isomorphism classes with $\mathrm{Tr}(j^*j) = \mathrm{Tr}(j_-^* j_-) + \mathrm{Tr}(j_+^* j_+) = \alpha_- + \beta_- + \alpha_+ + \beta_+ = 2$. Its peak is the abelian Lie algebra $\mathbb{R}^6$. The following picture illustrates the set $\mathcal{N}_6^0$, where the invariant t is omitted over the interior of $\mathcal{N}_{6,2}$. We have chosen a fundamental domain for the τ-action such that the parameters $\alpha_\pm, \beta_\pm, t$ always satisfy $\beta_- - \alpha_- \geq \beta_+ - \alpha_+$. We then only need to identify $(\alpha_\pm, \beta_\pm, t) \sim (\alpha_\mp, \beta_\mp, t)$ if $\beta_- - \alpha_- = \beta_+ - \alpha_+$. On the right hand face of $\mathcal{N}_{6,2}$ this requires to identify the two triangles by the reflection indicated by the two arrows $\leftrightarrow$. The dots $\bullet$ mark standard representatives for the seven different isomorphism classes of Lie algebras and are identified in the next section.

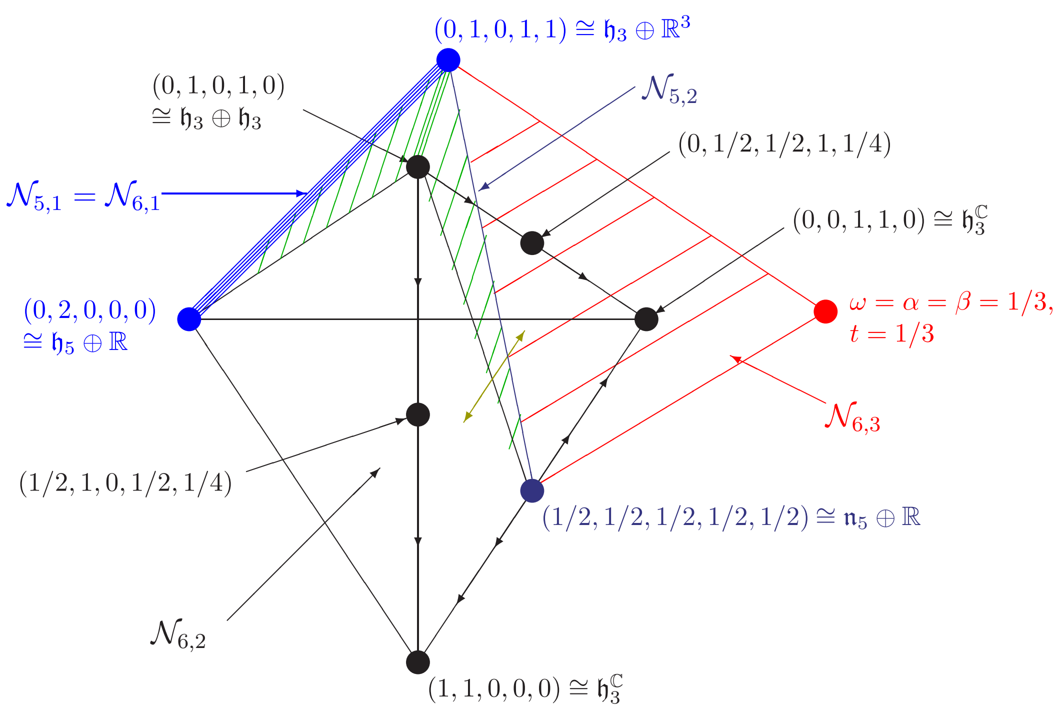

FIGURE 1. $\mathcal{N}_6^0$

4.2. **Classification of 6-dimensional 2-step nilpotent Lie algebras.** Next, we determine the isomorphism classes (disregarding the metric) of 2-step nilpotent 6-dimensional Lie algebras and exhibit canonical representatives. To this end, we compute the action of $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(4,\mathbb{R})$ on our invariants.

Remark 2. Using the $\mathrm{GL}(2,\mathbb{R})$-action for any j we find an equivalent one which is an isometric monomorphism.

Remark 3. For isometric j, we can simultaneously diagonalize $j_+^* j_+$ and $j_-^* j_-$. For $\mathcal{N}_{6,2}$, this yields the relations $\alpha_- + \beta_+ = 1 = \alpha_+ + \beta_-$ and $t = \alpha_-\beta_+ + \alpha_+\beta_-$. For $\mathcal{N}_{6,3}$, we have $\omega = \alpha = \beta = \frac{1}{2}$ and for $\mathcal{N}_{6,1}$ we get $\alpha_- = \alpha_+ = 0$, $\beta_+ + \beta_- = 1$ and $t = \beta_+\beta_-$.

A pair $(T, S) \in \mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(4,\mathbb{R})$ acts on j by replacing $j(z)$ with $S^* j(Tz) S$. In the bases $e_i^\pm$ for $\mathbb{R}^3_\pm = \mathfrak{so}_\pm(3) \subset \mathfrak{so}(4)$ and (e_1, e_2) for $\mathbb{R}^2$, we can put j in the form

$$j = \begin{pmatrix} a_- & 0 \\ 0 & b_- \\ 0 & 0 \\ p & r \\ 0 & q \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

with $0 \le a_- \le b_-$, $0 \le p, q, r$, using only the $\mathrm{O}(2) \times \mathrm{O}(4) \subset \mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(4,\mathbb{R})$ action. The coefficients $a_-, b_-, p, q, r \in \mathbb{R}_0^+$ are determined from the invariants by solving the equations

$$\begin{gathered} a_-^2 = \alpha_-, \quad b_-^2 = \beta_-, \quad p^2 + q^2 + r^2 = \alpha_+ + \beta_+, \\ p^2 q^2 = \alpha_+\beta_+, \quad \alpha_- p^2 + \beta_-(q^2 + r^2) = t. \end{gathered}$$

We first assume that j is isometric, i.e. $r = 0$, $a_-^2 + p^2 = 1 = b_-^2 + q^2$ and $p^2 = \beta_+$, $q^2 = \alpha_+$. Possibly interchanging $\pm$ we may also suppose $\beta_- \ge \beta_+$. Then, the only free invariants for this case are $\alpha_\pm$ and satisfy the conditions $\alpha_- \ge \alpha_+$ and $\alpha_+ \le 1 - \alpha_-$.

By means of the isomorphism (4.3) j defines the homomorphism $j\colon \mathbb{R}^2 \to \mathfrak{so}(4)$ given by

$$j(u, v) = \begin{pmatrix} 0 & (a_- + p)u & (b_- + q)v & 0 \\ -(a_- + p)u & 0 & 0 & (b_- - q)v \\ (-b_- - q)v & 0 & 0 & (-a_- + p)u \\ 0 & (q - b_-)v & (a_- - p)u & 0 \end{pmatrix}$$

In case $b_- - q \neq 0$, this is equivalent to the matrix with coefficients $(a_-, b'_-, p, 0, 0)$, $b'_- = \sqrt{(b_- + q)(b_- - q)}$, via the matrix

$$S := \begin{pmatrix} \lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

with $\lambda = \left(\dfrac{b_- + q}{b_- - q}\right)^{1/4}$. By rescaling v we can keep j isometric. Similarly, in case $a_- - p > 0$, we use a matrix

$$T := \begin{pmatrix} \lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

with $\lambda = \left(\dfrac{a_- + p}{a_- - p}\right)^{1/4}$ to see that any j with coefficients $(a_-, b_-, p, q, 0)$ is equivalent to one with coefficients $(a'_-, b_-, 0, q, 0)$ where $a'_- = \sqrt{(a_- + p)(a_- - p)}$.
In case $a_- - p < 0$ we can replace $(a_-, b_-, p, q, 0)$ by $(0, b_-, p, q, 0)$ by means of the above matrix T with $\lambda = \left(\frac{a_- + p}{p - a_-}\right)^{1/4}$. In order to keep j isometric, we rescale u.

The diagram below visualizes the subset of $\mathcal{N}_{6,2}$ represented by isometric j. With respect to the action of $\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(4, \mathbb{R})$ it decomposes into four isomorphism classes of Lie algebras indicated by the components in the picture.

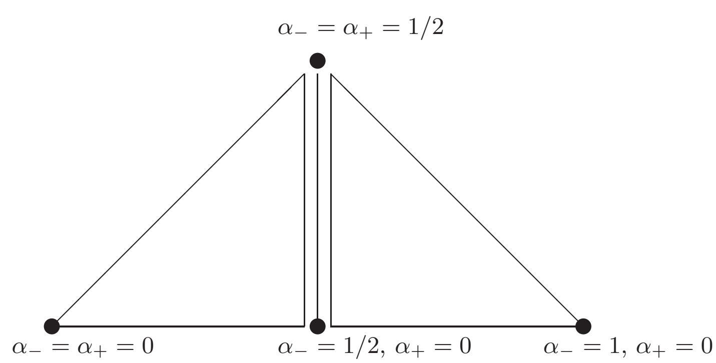

FIGURE 2. $\mathcal{D}_{6,2}$

Remark 4. If j is not isometric and has invariants $(\alpha_{\mp}, \beta_{\mp}, t)$ we first compute the coefficients (a_-, b_-, p, q, r) to write j in the shape (4.4). With

$$A := \begin{pmatrix} 1 & \dfrac{-pr}{(a_-^2 + p^2)} \\ 0 & 1 \end{pmatrix} .$$

and $B = \mathrm{diag}(1/\|jAe_1\|, 1/\|jAe_2\|)$ we get that jAB is isometric. Computing $\alpha_\pm(jAB)$, the isomorphism type of j can be determined.

In $\mathcal{N}_6$ we get the following isomorphism types, where the parameters are given for isometric j.

(1) Any Lie algebra in $\mathcal{N}_{6,1}$ is isomorphic to a Lie algebra with parameters $\alpha_\pm = 0 = \beta_+,\ \beta_- = 1,\ t = 0$ or $\alpha_\pm = 0, \beta_+ = \beta_- = 1/2,\ t = 1/4$. The first type is the product $(0,0,0,0,0,12+34) = \mathbb{R}^3 \oplus \mathfrak{h}_3$. The second type is a product $(0,0,0,0,0,12) = \mathfrak{h}_5 \oplus \mathbb{R}$.

(2) In $\mathcal{N}_{6,2}$ we have four isomorphism types corresponding to
(a) $\alpha_\pm < 1/2$, which gives $(0,0,0,0,12,34) = \mathfrak{h}_3 \oplus \mathfrak{h}_3$
(b) $\alpha_+ < 1/2 = \alpha_-$, $(0,0,0,0,12,14+23)$
(c) $\alpha_+ = 1/2 = \alpha_-$, $(0,0,0,0,12,13) = \mathfrak{n}_5 \oplus \mathbb{R}$ with $\mathfrak{n}_5 \in \mathcal{N}_{5,2}$ the unique non trivial isomorphism type
(d) $\alpha_+ < 1/2 < \alpha_-$, $(0,0,0,0,13+42,14+23) = \mathfrak{h}_3^{\mathbb{C}}$.

(3) Any Lie algebra in $\mathcal{N}_{6,3}$ is isomorphic to $(0,0,0,12,13,23)$ with $\omega = \alpha = \beta = 1/2$.

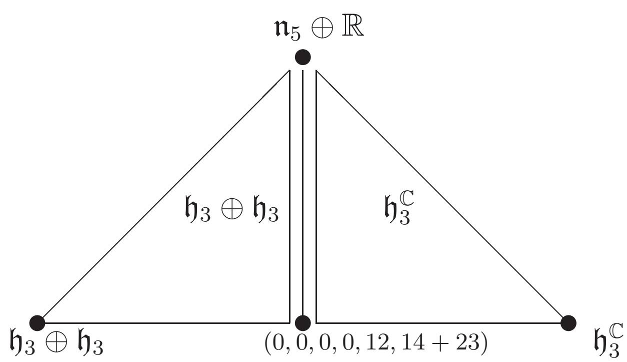

FIGURE 3. Isomorphism classes in $\mathcal{D}_{6,2}$

Remark 5. A Lie algebra $c \in \widehat{\mathcal{L}}_n \subset \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ is said to degenerate to another Lie algebra $\tilde{c}$, if $\tilde{c}$ is

represented by a structure which lies in the Zariski closure of the $\mathrm{GL}(n,\mathbb{R})$-orbit of a structure which represents c. In this case the entire $\mathrm{GL}(n,\mathbb{R})$-orbit of $\tilde{c}$ in $\widehat{\mathcal{L}}_n$ lies in the closure of the orbit of c [1, 12]. Recall that c degenerates to $\tilde{c}$ if there exist $g_s \in \mathrm{GL}(n,\mathbb{R})$ such that $\lim_{s\to 0} g_s \cdot c = \tilde{c}$. Using this, it is easy to see that the Lie algebras $\mathfrak{h}_3^{\mathbb{C}}, \mathfrak{h}_3 \oplus \mathfrak{h}_3,\ (0,0,0,0,12,14+23)$ all degenerate to $\mathfrak{n}_5 \oplus \mathbb{R}$ (the top point in Figure 3).

Remark 6. Using Remark 4, one can determine the structure equations for any 6-dimensional 2-step nilpotent Lie algebra. As an example, from the isomorphism

$$\mathfrak{so}(4) \cong \Lambda^2\mathbb{R}^4 \tag{4.5}$$

we will give the structure equations for the Lie algebras in $\mathcal{D}_{6,2}$.

Indeed, if one fixes a non-zero element $w \in \Lambda^4\mathbb{R}^4$, one can consider the bilinear form ϕ of signature (3,3) on $\Lambda^2\mathbb{R}^4$ defined by $\sigma \wedge \tau = \phi(\sigma,\tau)w$.

Given an orientation and a metric g on $\mathbb{R}^4$ (and so on $\Lambda^2\mathbb{R}^4$), there is an $SO(4)$-decomposition

$$\Lambda^2\mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{4.6}$$

where $\Lambda^2_\pm$ are the eigenspaces of the conformally invariant involution $*$ of $\Lambda^2\mathbb{R}^4$ for which $\phi(*\sigma,\tau) = g(\sigma,\tau)$. From a representation-theoretic point of view, (4.6) is equivalent to the Lie algebra splitting (4.3).
If one chooses a basis $\{e^1,e^2,e^3,e^4\}$ of $\mathbb{R}^4$ such that $w = e^1\wedge e^2\wedge e^3\wedge e^4$, then

$$\begin{aligned}
\Lambda^2_+ &= \operatorname{span}\{e^1\wedge e^2 + e^3\wedge e^4, e^1\wedge e^3 + e^4\wedge e^2, e^1\wedge e^4 + e^2\wedge e^3\},\\
\Lambda^2_- &= \operatorname{span}\{e^1\wedge e^2 - e^3\wedge e^4, e^1\wedge e^3 - e^4\wedge e^2, e^1\wedge e^4 - e^2\wedge e^3\}.
\end{aligned}$$

Using (4.5) and the embeddings (4.3) one has the following identifications

$$\begin{aligned}
e_1^+ &\sim e^1\wedge e^2 + e^3\wedge e^4,\\
e_2^+ &\sim e^1\wedge e^3 - e^2\wedge e^4,\\
e_1^- &\sim e^1\wedge e^2 - e^3\wedge e^4,\\
e_2^- &\sim e^1\wedge e^3 + e^2\wedge e^4
\end{aligned}$$

and thus $\mathfrak{n}_{(\alpha_+,\alpha_-)}$ has structure equations

$$\begin{aligned}
de^i &= 0, \qquad i = 1,\dots,4\ ,\\
de^5 &= (a_- + p)\, e^1\wedge e^2 + (a_- - p)\, e^3\wedge e^4,\\
de^6 &= (b_- + q)\, e^1\wedge e^3 - (b_- - q)\, e^2\wedge e^4,
\end{aligned} \tag{4.7}$$

where

$$a_- = \sqrt{\alpha_-}, \quad b_- = \sqrt{\beta_-}, \quad p = \sqrt{\beta_+}, \quad q = \sqrt{\alpha_+}.$$

Remark 7. Next, we compute the infinitesimal rank of a Lie algebra in $\mathcal{D}_{6,2}$.

The rank of a geodesic in a Riemannian manifold M is the dimension of the real vector space of parallel Jacobi fields along it. The rank

$\operatorname{rk}(M)$ of M is the minimum of the ranks of all its geodesics. Recall that the Jacobi-operator R_v is the endomorphism of T_pM given by $w \mapsto R_{v,w}v$. The infinitesimal rank

infrk(M) of M is the minimal dimension of the kernels of its Jacobi-operators [15]. A Riemannian manifold M has higher (infinitesimal) rank if (inf)rk$(M) \geq 2$.

First, we use the structure equations (4.7) to compute the curvature tensor with respect to the metric g for which the forms (e^i) are dual to an orthonormal basis (e_i).

The non vanishing components $R_{ijhk} = g(R_{e_i,e_j}e_h, e_k)$ of the Riemannian curvature tensor are:

$$
\begin{aligned}
&R_{1212} = -\tfrac{3}{4}(a_- + p)^2,\\
&R_{1234} = -\tfrac{1}{2}a_-^2 + \tfrac{1}{2}p^2 + \tfrac{1}{4}b_-^2 - \tfrac{1}{4}q^2 = R_{3412},\\
&R_{1313} = -\tfrac{3}{4}(b_- - q)^2,\\
&R_{1324} = -\tfrac{1}{4}a_-^2 + \tfrac{1}{4}p^2 + \tfrac{1}{2}b_-^2 - \tfrac{1}{2}q^2 = R_{2413},\\
&R_{1423} = \tfrac{1}{4}a_-^2 - \tfrac{1}{4}p^2 + \tfrac{1}{4}b_-^2 - \tfrac{1}{4}q^2 = R_{2314},\\
&R_{1456} = \tfrac{1}{2}pq - \tfrac{1}{2}a_-b_- = R_{5614},\\
&R_{1515} = \tfrac{1}{4}(a_- + p)^2 = R_{2525},\\
&R_{1546} = -\tfrac{1}{4}(b_- + q)(a_- - p) = R_{4615},\\
&R_{1616} = \tfrac{1}{4}(b_- + q)^2 = R_{3636},\\
&R_{1645} = \tfrac{1}{4}(b_- - q)(a_- + p) = R_{4516},\\
&R_{2356} = -\tfrac{1}{2}a_-b_- - \tfrac{1}{2}pq = R_{5623},\\
&R_{2424} = -\tfrac{3}{4}(b_- - q)^2,\\
&R_{2536} = -\tfrac{1}{4}(a_- - p)(b_- - q) = R_{3625},\\
&R_{2626} = \tfrac{1}{4}(b_- - q)^2 = R_{4646},\\
&R_{2635} = \tfrac{1}{4}(b_- + q)(a_- + p) = R_{3526},\\
&R_{3434} = -\tfrac{3}{4}(a_- - p)^2,\\
&R_{3535} = \tfrac{1}{4}(a_- - p)^2 = R_{4545}.
\end{aligned}
$$

The infinitesimal rank is 1 for any a_-, p, b_-, q (with respect to the above metric), except for $(a_-, b_-) = (1, 0)$ and $(a_-, b_-) = (\sqrt{2}/2, \sqrt{2}/2)$. Indeed the Jacobi operator:

$$R_{e_1+e_6} : X \mapsto R_{e_1+e_6,X}e_1 + e_6$$

whose associated matrix is

$$
\begin{pmatrix}
\frac{1}{4}\eta^2 & 0 & 0 & 0 & 0 & -\frac{1}{4}\eta^2\\
0 & -\frac{1}{4}[3\nu^2 - (b_- - q)^2] & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{1}{2}\eta^2 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{4}(b_- - q)^2 & -\rho & 0\\
0 & 0 & 0 & \rho & \frac{1}{4}\nu^2 & 0\\
-\frac{1}{4}\eta^2 & 0 & 0 & 0 & 0 & \frac{1}{4}\eta^2
\end{pmatrix}
$$

(with $\nu = a_- + p$, $\eta = b_- + q$, $\rho = \frac{1}{4}[3pq - 3a_-b_- + pb_- - a_-q]$) has one dimensional kernel, except for the following cases:

(1) $a_- = q$, $b_- = p$;
(2) $a_- = p = \sqrt{2}/2$.

If one considers, in addition, the Jacobi operator:

$$R_{e_2+e_5} : X \mapsto R_{e_2+e_5,X} e_2 + e_5$$

its associated matrix is

$$\begin{pmatrix} -\frac{1}{2}\nu^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4}\nu^2 & 0 & 0 & 0 & -\frac{1}{4}\nu^2 \\ 0 & 0 & \frac{1}{4}(a_- - p)^2 & 0 & -\zeta & 0 \\ 0 & 0 & 0 & -\frac{1}{4}[3(b_- - q)^2 - (a_- - p)] & 0 & 0 \\ 0 & -\frac{1}{4}\nu^2 & 0 & 0 & 0 & \frac{1}{4}\nu^2 \\ 0 & 0 & -\zeta & 0 & \frac{1}{4}(b_- - q) & 0 \end{pmatrix},$$

(with $\zeta = \frac{1}{4}[3a_-b_- + 3pq + a_-q + pb_-]$). Again, the dimension of the kernel of $R_{e_2+e_5}$ is generically one. Both $R_{e_1+e_6}$ and $R_{e_2+e_5}$ have kernel of dimension bigger than one if $b_- = 0 = p$ and $a_- = b_- = \sqrt{2}/2$. These two cases correspond to the Lie algebras $\mathfrak{n}_{(1,1)} \cong \mathfrak{h}_3 \oplus \mathfrak{h}_3$ and $\mathfrak{n}_(1/2, 1/2) \cong \mathfrak{n}_5 \oplus \mathbb{R}$, which are both Riemannian products of (infinitesimal) rank one Lie algebras. Thus, their (infinitesimal) rank is two.

Acknowledgments We are grateful to Simon Salamon and Sergio Garbiero for helpful discussions and useful references.

References

[1] D. Burde, Degenerations of nilpotent Lie algebras. J. Lie Theory **9** (1999), no. 1, 193–202.

[2] I. Dotti, A. Fino, HyperKähler torsion structures invariant by nilpotent Lie groups, Classical Quantum Gravity **19** (2002), no. 3, 551–562.

[3] I. Dotti, A. Fino: Abelian hypercomplex 8-dimensional nilmanifolds, *Ann. Glob. Anal. and Geom.* **18** (2000), 47-59.

[4] P. Eberlein, Geometry of 2-step nilpotent groups with a left invariant metric, Ann. Scient. Ec. Norm. Sup., 4, **27** (1994), 611-660.

[5] P. Eberlein, The moduli space of 2-step nilpotent Lie groups of type (p,q), preprint (2002).

[6] P. Eberlein, Geometry of 2-step nilpotent Lie groups ", preprint (2003).

[7] C. Gordon, Y. Mao, D. Schueth, Symplectic rigidity of geodesic flows on two-step nilmanifolds. Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 4, 417–427.

[8] M. Goze, Y. Khakimdjanov, Nilpotent Lie algebras, Mathematics and its Applications, 361, Kluwer Academic Publishers Group, Dordrecht, 1996.

[9] A. Kaplan, Riemannian nilmanifolds attached to Clifford modules, Geometriae Dedicata **11** (1981), 127-136.

[10] A. Kaplan, On the geometry of groups of Heisenberg type, Bull. London Math. Soc. **15** (1983), 35-42.

[11] J. Lauret, Homogeneous nilmanifolds attached to representations of compact Lie groups, Manuscripta math. **99** (1999), 287-309.

[12] J. Lauret, Degenerations of Lie algebras and geometry of Lie groups, Differential Geometry and its Applications **18** (2003), 177-194

[13] L. Magnin, Sur les algèbres de Lie nilpotentes de dimension ≤ 7, J. Geom. Phys. **3** (1986), no. 1, 119–144.
[14] S. Salamon, Complex structures on nilpotent Lie algebras, J. Pure Appl. Algebra **157** (2001), no. 2-3, 311-333.
[15] E. Samiou, 2-step nilpotent Lie groups of higher rank, Manuscripta math. **107** (2002), 101-110.

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